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2010 J. Phys. A: Math. Theor. 43 095202

(http://iopscience.iop.org/1751-8121/43/9/095202)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 095202 (15pp) doi:10.1088/1751-8113/43/9/095202

Functional differential equations for the q-Fourier transform of q-Gaussians

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Received 8 September 2009, in final form 7 January 2010
Published 15 February 2010
Online at stacks.iop.org/JPhysA/43/095202

Abstract
In this paper the question 'is the q-Fourier transform of a q-Gaussian a q'-Gaussian (with some q') up to a constant factor?' is studied for the whole range of $q \in (-\infty, 3)$. This question is connected with applicability of the q-Fourier transform in the study of limit processes in nonextensive statistical mechanics. Using the functional differential equation approach we prove that the answer is affirmative if and only if $1 \leqslant q < 3$, excluding two particular cases of $q < 1$, namely $q = \frac{1}{2}$ and $q = \frac{2}{3}$. Complementarily, we discuss some applications of the q-Fourier transform to nonlinear partial differential equations such as the porous medium equation.

PACS numbers: 02.30.Sa, 02.30.Uu, 89.65.Gh, 89.75.Da
Mathematics Subject Classification: 35Q84, 37A50, 43A32, 65L03

1. Introduction

Approximately a century after Boltzmann's seminal works which have turned into the cornerstones of statistical mechanics, Tsallis [1] introduced an entropic form aimed to accommodate the description of systems whose fundamental features may not be fitted in the Boltzmann–Gibbs formalism (see details in [2–4]). Tsallis' entropic form, which is usually called the non-additive q-entropy, recovers the classic Boltzmann–Gibbs entropic form, $S(f) = -\int f(x) \ln f(x)\, \mathrm{d}x$ in the limit case $q \to 1$. Concomitantly, there is the nonextensive statistical mechanics formalism based on q-algebra and the q-Gaussian probability density function, which maximizes q-entropy under certain appropriate constraints (see [1, 5] and references therein). Recently, the q-Fourier transform [6] was introduced as a tool for the study of attractors of strongly correlated random variables in conjunction with the q-central limit theorem. The existence of such a theorem within nonextensive statistical mechanics was first conjectured in [7, 8]. In this paper we shed light on the question—whether the q-Fourier transform of a q-Gaussian is a q'-Gaussian, clarifying thereupon applicability of the q-Fourier transform technique as a mathematical tool. A key to this matter is crucial because the q-Fourier transform is relevant to the study of limit distributions of strongly correlated random variables, as well as to solutions of partial differential equations with physical significance. Moreover, a positive answer implies validating a mapping relation of q onto q' obtained from the q-Fourier transform. This relation has been predominant for the establishment of other stable distributions, namely the (q, α)-stable distributions [9]. We recall that, by definition, the *q-Fourier transform* of a nonnegative $f \in L_1(R)$ is defined by the formula

$$F_q[f](\xi) = \int_{\mathrm{supp}\, f} e_q^{\mathrm{i}x\xi} \otimes_q f(x)\,\mathrm{d}x, \qquad \xi \in (-\infty, \infty), \tag{1}$$

where $q < 3$, the symbol $\otimes_q$ stands for the q-product and

$$e_q^z = (1 + (1-q)z)^{1/(1-q)}, \qquad z \in C, \tag{2}$$

is a q-exponential, which is the usual exponential function e^z in the limit $q \to 1$, and defined for all $z \in C \backslash z_0$, $z_0 = -1/(1-q)$, with principal values along the cut $(-\infty, z_0)$ if $q \neq 1$ (see [6, 7] for details). The equality

$$e_q^{\mathrm{i}x\xi} \otimes_q f(x) = f(x) e_q^{\frac{\mathrm{i}x\xi}{[f(x)]^{1-q}}},$$

valid for all $x \in \mathrm{supp}\, f$ implies the following representation for the q-Fourier transform without usage of the q-product:

$$F_q[f](\xi) = \int_{\mathrm{supp}\, f} f(x) e_q^{\mathrm{i}x\xi[f(x)]^{q-1}}\,\mathrm{d}x. \tag{3}$$

The paper is organized as follows: in section 2 we mention some properties of the q-Fourier transform. In section 3 we derive functional differential equations for the q-Fourier transform of q-Gaussians. Then, based on the results of this section, we show that the answer to the above question is affirmative for all $1 \leqslant q < 3$, and for two particular values of $q < 1$, namely for $q = 1/2$ and $q = 2/3$. We also show that if $q < 1$, except two values mentioned above, the q-Fourier transform of a q-Gaussian is no longer a q'-Gaussian, $\forall q' < 3$. A relevant physical application of the q-Fourier transform and the functional differential equations studied in section 3 is addressed in section 4.

2. Preliminaries

Representation (3) for the q-Fourier transform implies the following proposition.

Proposition 2.1. *For any constants $a > 0$, $b \neq 0$,*

(i) $F_q[af(x)](\xi) = aF_q[f(x)]\left(\frac{\xi}{a^{1-q}}\right)$;
(ii) $F_q[f(bx)](\xi) = \frac{1}{b} F_q[f(x)]\left(\frac{\xi}{b}\right)$.

Let β be a positive number. By definition, the function

$$G_q(\beta; x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}, \tag{4}$$

with domains given below, is called a *q-Gaussian density function*:

(i) if $q < 1$, then $G_q(\beta; x)$ is defined on the compact set $[-K_\beta, K_\beta]$, where $K_\beta = (\beta(1-q))^{-1/2}$;
(ii) if $1 \leqslant q < 3$, then $G_q(\beta; x)$ is defined on the whole real axis $\mathbb{R} = (-\infty, \infty)$.

In expression (4) C_q is the normalizing constant, i.e. $C_q = \int_{-\infty}^{\infty} e_q^{-x^2}\,\mathrm{d}x$, whose explicit form is given by (see, e.g., [6])

$$C_q = \begin{cases} \dfrac{2\sqrt{\pi}\,\Gamma\left(\frac{1}{1-q}\right)}{(3-q)\sqrt{1-q}\,\Gamma\left(\frac{3-q}{2(1-q)}\right)}, & -\infty < q < 1, \\ \sqrt{\pi}, & q = 1, \\ \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\,\Gamma\left(\frac{1}{q-1}\right)}, & 1 < q < 3. \end{cases} \tag{5}$$

We use the convention $K_\beta = \infty$ if $q \geqslant 1$, since, by definition, the support of the q-Gaussian is not bounded in this case.

Note that q-exponentials possess the property $e_q^z \otimes_q e_q^w = e_q^{z+w}$ [12, 13]. This immediately implies the following assertion.

Proposition 2.2. *For all $q < 3$ the q-Fourier transform of $e_q^{-\beta x^2}$, $\beta > 0$, can be written in the form*

$$F_q\left[e_q^{-\beta x^2}\right](\xi) = \int_{-K_\beta}^{K_\beta} e_q^{-\beta x^2 + \mathrm{i}x\xi}\,\mathrm{d}x. \tag{6}$$

Corollary 2.3. *Let $q < 3$. Then*

$$F_q\left[e_q^{-\beta x^2}\right](\xi) = 2\int_0^{K_\beta} e_q^{-\beta x^2} \cos_q\left(\frac{x\xi}{[e_q^{-\beta x^2}]^{1-q}}\right)\mathrm{d}x, \qquad \forall\, q,$$

where

$$\cos_q(x) = \frac{e_q^{\mathrm{i}x} + e_q^{-\mathrm{i}x}}{2}.$$

The assertion below was proved in [6].

Proposition 2.4. *Let $1 \leqslant q < 3$. Then*

$$F_q[G_q(\beta; x)](\xi) = e_{q_1}^{-\beta_* \xi^2}, \qquad \xi \in \mathbb{R}, \tag{7}$$

where $q_1 = \frac{1+q}{3-q}$ and $\beta_* = \frac{3-q}{8\beta^{2-q} C_q^{2(q-1)}}$.

Proposition 2.5. *Let $q < 1$. Then*

$$F_q[G_q(\beta, x)](\xi) = e_{q_1}^{-\beta_* \xi^2}\left(1 - \frac{2}{C_q}\,\mathrm{Im}\int_0^{d_\xi} e_q^{b_\xi + \mathrm{i}\tau}\,\mathrm{d}\tau\right), \qquad \xi \in \left(-K_{\frac{1}{4\beta}}, K_{\frac{1}{4\beta}}\right),$$

where q_1 and β_* are as in proposition 2.4 and $b_\xi + \mathrm{i}d_\xi = \dfrac{K_\beta\sqrt{\beta} - \mathrm{i}\frac{\xi}{2\sqrt{\beta}}}{\left[e_q^{-\frac{\xi^2}{4\beta}}\right]^{\frac{1-q}{2}}}$.

Proof. The proof of this statement can be obtained applying the Cauchy theorem, that is by integrating the function $e_q^{-\beta z^2 + \mathrm{i}z\xi}$ over the closed contour $C = C_0 \cup C_1 \cup C_- \cup C_+$, where $C_p = (-K_\beta + p\mathrm{i}, K_\beta + \mathrm{i}p)$, $p = 0, 1$, and $C_\pm = [\pm K_\beta, \pm K_\beta + \mathrm{i}]$. □

Unifying propositions 2.4 and 2.5,

$$F_q[G_q(\beta, x)](\xi) = e_{q_1}^{-\beta_* \xi^2} + I_{(-\infty,0)}(q) T_q(\xi),$$

where $I_{(a,b)}(\cdot)$ designates the indicator function of an interval (a, b), and

$$T_q(\xi) = -\frac{2}{C_q} e_{q_1}^{-\beta_* \xi^2} \operatorname{Im} \int_0^{d_\xi} e_q^{b_\xi + i\tau} \, d\tau.$$

Thus, for $q \geqslant 1$, the operator F_q transforms a q-Gaussian into a q_1-Gaussian with the factor $C_{q_1}\beta^{-1/2}$. However, for $q < 1$ the additional tail $T_q(\xi)$ appears.

Further, introduce a sequence q_n defined as

$$q_n = \frac{2q - n(q-1)}{2 - n(q-1)}, \tag{8}$$

where $-\infty < n < \frac{2}{q-1} - 1$ if $1 < q < 3$, and $n > -\frac{2}{1-q}$ if $q < 1$. Obviously, $q_0 = q$. Note also that if $q = 1$, then $q_n = 1$ for all $n = 0, \pm 1, \ldots$. Let $\mathbb{Z}$ be the set of all integer numbers. Denote by $\mathbb{N}_q$ a subset of $\mathbb{Z}$ defined as

$$\mathbb{N}_q = \begin{cases} \left\{n \in \mathbb{Z} : n < \frac{2}{q-1} - 1\right\}, & \text{if} \quad 1 < q < 3, \\ \mathbb{Z}, & \text{if} \quad q = 1, \\ \left\{n \in \mathbb{Z} : n > -\frac{2}{1-q}\right\}, & \text{if} \quad q < 1. \end{cases}$$

Proposition 2.6. *For all $n \in \mathbb{N}_q$ the following relations hold:*

(i) $(3 - q_n)q_{n+1} = (3 - q_{n-2})q_n$,

(ii) $2C_{q_{n-2}} = \sqrt{q_n}\,(3 - q_n)\,C_{q_n}$.

Proof.

(i) It follows from the definition of q_n that $q_{n+1} = (1 + q_n)/(3 - q_n)$. This yields

$$(3 - q_n)q_{n+1} = 1 + q_n = \left(1 + \frac{1}{q_n}\right) q_n. \tag{9}$$

Further, it is easy to verify that the equality $q_{k-1} + q_{k+1}^{-1} = 2$ holds for all $k \in \mathbb{N}_q$. Applying this relationship for $k = n - 1$, we have $1/q_n = 2 - q_{n-2}$. Now taking this into account in (9), we obtain (i).

(ii) Obviously, for $q = 1$ relationship (ii) reads $2\sqrt{\pi} = 2\sqrt{\pi}$. Let $q \neq 1$. Note that for any $n \in \mathbb{N}_q$ the condition $1 < q < 3$ implies $1 < q_n < 3$, as well as the condition $q < 1$ implies $q_n < 1$. Using the explicit forms for C_q given in (5) and the relationship $2 - q_{n-2} = 1/q_n$, one obtains in the case $1 < q < 3$

$$\frac{2C_{n-2}}{C_n} = \frac{\sqrt{q_n}\,\Gamma\left(\frac{1+q_n}{2(q_n-1)}\right)}{\frac{1}{2(q_n-1)}\Gamma\left(\frac{3-q_n}{2(q_n-1)}\right)} = \sqrt{q_n}(3 - q_n);$$

and in the case $q < 1$

$$\frac{2C_{n-2}}{C_n} = \frac{\sqrt{q_n}(3 - q_n)}{\frac{1+q_n}{2(1-q_n)}} \frac{\Gamma\left(\frac{3-q_n}{2(1-q_n)}\right)}{\Gamma\left(\frac{1+q_n}{2(1-q_n)}\right)} = \sqrt{q_n}(3 - q_n),$$

completing the proof of part (ii). □

3. Main results

3.1. Functional differential equations

Let $g_q(\beta,\xi)$ be the q-Fourier transform of a q-Gaussian $G_q(\beta,\xi)$, i.e. $g_q(\beta,\xi) = F_q[G_q(\beta,x)](\xi)$, and $g_q(\xi) = g_q(1,\xi)$ for $\beta = 1$. Further, let $Y_q(\xi) = F_q\big[e_q^{-x^2}\big](\xi)$. In accordance with proposition 2.2,

$$Y_q(\xi) = \int_{-K}^{K} e_q^{-x^2+\mathrm{i}x\xi}\,\mathrm{d}x, \qquad \xi\in\mathbb{R}, \tag{10}$$

where $K = K_1 = \frac{1}{\sqrt{1-q}}$ if $q<1$, and $K=\infty$, if $q\geqslant 1$.

Lemma 3.1. *For any $q<3$ and $\beta>0$ we have*

(i) $g_q(\beta,\xi) = g_q\big(\frac{\xi}{(\sqrt{\beta})^{2-q}}\big)$;

(ii) $g_q(\xi) = \frac{1}{C_q}Y_q\big(C_q^{1-q}\xi\big)$.

Proof. The proof straightforwardly follows from the properties of the operator F_q indicated in proposition 2.1. □

These two formulas yield

$$F_q[G_q(\beta,x)](\xi) = \frac{1}{C_q}Y_q\left(\left(\frac{C_q}{\sqrt{\beta}}\right)^{1-q}\frac{\xi}{\sqrt{\beta}}\right). \tag{11}$$

Moreover, $g_q(\beta,0)=1$, which implies $g_q(0)=1$ and $Y_q(0)=C_q$. Thus, in order to know the properties of the q-Fourier transform of q-Gaussians it suffices to study $Y_q(\xi)$.

Theorem 3.2. *Let $1\leqslant q<3$ and q_n, $n\in\mathbb{N}_q$, are defined in equation (8). Then $Y_{q_n}(\xi)$ satisfies the following homogeneous functional differential equation:*

$$2\sqrt{q_n}\frac{\mathrm{d}Y_{q_n}(\xi)}{\mathrm{d}\xi} + \xi Y_{q_{n-2}}(\sqrt{q_n}\xi) = 0, \qquad \xi\in\mathbb{R}. \tag{12}$$

Proof. Differentiating $Y_q(\xi) = \int_{-K}^{K} e_q^{-x^2+\mathrm{i}x\xi}\,\mathrm{d}x$ with respect to ξ, we have

$$\frac{\mathrm{d}Y_q(\xi)}{\mathrm{d}\xi} = \mathrm{i}\int_{-K}^{K} x\big(e_q^{-x^2+\mathrm{i}x\xi}\big)^q\,\mathrm{d}x.$$

Further, integrating by parts,

$$\frac{\mathrm{d}Y_q(\xi)}{\mathrm{d}\xi} = \frac{-\mathrm{i}}{2}\int_{-K}^{K}\mathrm{d}\big(e_q^{-x^2+\mathrm{i}x\xi}\big) - \frac{\xi}{2}\int_{-K}^{K}\big(e_q^{-x^2+\mathrm{i}x\xi}\big)^q\,\mathrm{d}x. \tag{13}$$

Obviously, the first integral vanishes if $q\geqslant 1$. Further, using $\big(e_q^y\big)^q = e_{2-1/q}^{qy}$, which is valid for any $q<3$ (see [6]), the second integral can be represented in the form

$$\int_{-K}^{K}\big(e_q^{-x^2+\mathrm{i}x\xi}\big)^q\,\mathrm{d}x = \frac{1}{\sqrt{q}}\int_{-K}^{K} e_{2-1/q}^{-x^2+\mathrm{i}x\sqrt{q}\xi}\,\mathrm{d}x = \frac{1}{\sqrt{q}}Y_{2-1/q}(\sqrt{q}\xi). \tag{14}$$

Hence, for $q\geqslant 1$ the function $F_q\big[e_q^{-x^2}\big](\xi)$ satisfies the functional differential equation

$$2\sqrt{q}\frac{\mathrm{d}Y_q(\xi)}{\mathrm{d}\xi} + \xi Y_{2-1/q}(\sqrt{q}\xi) = 0. \tag{15}$$

Setting $q=q_n, n\in\mathbb{N}_q$, and taking into account the relation $2-1/q_n = q_{n-2}$, we obtain equation (12). □

Theorem 3.3. *Let* $0 < q < 1$ *and* $q \neq l/(l+1), l = 1, 2, \ldots$*. Then* $Y_{q_n}(\xi)$ *satisfies the following inhomogeneous functional differential equation:*

$$2\sqrt{q_n}\frac{\mathrm{d}Y_{q_n}(\xi)}{\mathrm{d}\xi} + \xi Y_{q_{n-2}}(\sqrt{q_n}\xi) = r_{q_n}\xi^{\frac{1}{1-q_n}}, \qquad \xi \in \mathbb{R}, \tag{16}$$

where

$$r_{q_n} = 2\sqrt{q_n}\sin\frac{\pi}{2(1-q_n)}(1-q_n)^{\frac{1}{2(1-q_n)}}. \tag{17}$$

Proof. Assume that $0 < q < 1$ and $q \neq \frac{l}{l+1}, l = 1, 2, \ldots$. In this case the first integral on the right-hand side of (13) does not vanish, and takes the form

$$\int_{-K}^{K} \mathrm{d}\big(e_q^{-x^2+\mathrm{i}x\xi}\big) = e_q^{-K^2+\mathrm{i}K\xi} - e_q^{-K^2-\mathrm{i}K\xi} = 2\mathrm{i}\,\mathrm{Im}\, e_q^{-K^2+\mathrm{i}K\xi}.$$

Since $\mathrm{supp}\, e_q^{-x^2} = [-K, K]$, one has $e_q^{-K^2} = 0$. Hence,

$$e_q^{-K^2+\mathrm{i}K\xi} = 0 \otimes_q e_q^{\mathrm{i}K\xi} = [\mathrm{i}(1-q)K\xi]^{\frac{1}{1-q}}.$$

Further, taking into account that $K = 1/\sqrt{1-q}$, one obtains

$$\mathrm{Im}[\mathrm{i}(1-q)K\xi]^{\frac{1}{1-q}} = (1-q)^{\frac{1}{2(1-q)}}\sin\frac{\pi}{2(1-q)}\xi^{\frac{1}{1-q}}.$$

Note that the second integral on the right-hand side of equation (13) is the same as in the case of $1 < q < 3$. Consequently, $F_q\big[e_q^{-x^2}\big](\xi)$ satisfies the functional differential equation

$$2\sqrt{q}\frac{\mathrm{d}Y_q(\xi)}{\partial\xi} + \xi Y_{2-1/q}(\sqrt{q}\xi) = r_q\xi^{\frac{1}{1-q}}, \tag{18}$$

where

$$r_q = 2\sqrt{q}(1-q)^{\frac{1}{2(1-q)}}\sin\frac{\pi}{2(1-q)}.$$

Again, setting $q = q_n, n \in \mathbb{N}_q$, we arrive at the functional differential equation (16). □

Remark 3.4.

(i) If $q = 0$, then it is readily seen that $Y_0(\xi) = F_0\big[e_0^{-x^2}\big](\xi) = \int_{-1}^{1}(1 - x^2 + \mathrm{i}x\xi)\,\mathrm{d}x = 4/3$ for all $\xi \in \mathbb{R}$. Obviously, such $Y_0(\xi)$ cannot be a q'-Gaussian for any q'.

(ii) We will show later that a q-Fourier image of any q-Gaussian with $q < 0$ cannot be a function of the form $ae_{q'}^{-\beta\xi^2}$, for any $q' \in (-\infty, 3)$ (see theorem 3.16).

Let us now consider the cases $q = \ell/(\ell+1)$, $\ell = 1, 2, \ldots$, excluded from theorem 3.3. For these values of q we have $K = \sqrt{\ell+1}$ and

$$Y_q(\xi) = F_q\big[e_q^{-x^2}\big](\xi) = \int_{-\sqrt{\ell+1}}^{\sqrt{\ell+1}}\left(1 - \frac{1}{\ell+1}x^2 + \frac{1}{\ell+1}\mathrm{i}x\xi\right)^{\ell+1}\mathrm{d}x.$$

The latter is a polynomial of order ℓ if ℓ is even, and of order $\ell+1$ if ℓ is odd[3]. In order to reflect this fact we use the conventional notation $P_{\ell+1}(\xi) = Y_{\ell/(\ell+1)}(\xi)$ indicating the dependence on ℓ. Further, obviously $2 - \frac{1}{q} = \frac{\ell-1}{\ell}$. Consequently,

$$Y_{2-1/q}(\xi) = \int_{-\sqrt{\ell}}^{\sqrt{\ell}}\left(1 - \frac{1}{\ell}x^2 + \frac{1}{\ell}\mathrm{i}x\xi\right)^{\ell}\mathrm{d}x = P_\ell(\xi),$$

[3] P_ℓ does not contain odd-order terms.

and we note that $P_\ell(\xi)$ is a polynomial of order ℓ if ℓ is even, and of order $\ell - 1$ if ℓ is odd. Moreover, $P_\ell(\xi)$ is a symmetric function of ξ and $P_\ell(0) = C_{\frac{\ell-1}{\ell}} > 0$. Let ρ be a root of $P_\ell(\xi)$ closest to the origin. We will consider $P_\ell(\xi)$ only on its positivity interval $(-\rho, \rho)$.

Theorem 3.5. *Let $q = \frac{2m-1}{2m}$, $m = 1, 2, \ldots$. Then $Y_q(\xi)$ satisfies equation (12).*

Proof. Assume $\ell + 1 = 2m$, $m = 1, 2, \ldots$. In this case $Y_q(\xi) = P_{2m}(\xi)$ is a polynomial of order $2m$ and $Y_{2-1/q}(\xi) = P_{2m-1}(\xi)$ is a polynomial of order $2m - 2$. Moreover, it is easy to check $r_q = 0$. Thus, $Y_q(\xi)$ satisfies the consistent equation

$$2\sqrt{q}\frac{\mathrm{d}Y_q(\xi)}{\mathrm{d}\xi} + \xi Y_{2-1/q}(\sqrt{q}\xi) = 0, \qquad \xi \in \mathbb{R}. \tag{19}$$

□

Theorem 3.6. *Let $q = \frac{2m}{2m+1}$, $m = 1, 2, \ldots$. Then $Y_q(\xi)$ satisfies neither equation (12) nor (16).*

Proof. Let $\ell = 2m$, $m = 1, 2, \ldots$. Then $Y_q(\xi) = P_{2m+1}(\xi)$ is a polynomial of order $2m$, and so is $Y_{2-1/q}(\xi) = P_{2m}(\xi)$. Assume that $Y_q(\xi)$ satisfies equation (12), which in this particular case takes the form

$$2\sqrt{q}\frac{\mathrm{d}Y_q(\xi)}{\mathrm{d}\xi} + \xi P_{2m}(\xi) = 0. \tag{20}$$

Equation (20) is clearly inconsistent, since the derivative of a polynomial of order $2m$ cannot be a polynomial of order $2m + 1$. Analogously, $Y_q(\xi)$ cannot satisfy equation (16) either. Indeed, if $Y_q(\xi)$ solves equation (16), then in this particular case the equation would read

$$2\sqrt{q}\frac{\mathrm{d}Y_q(\xi)}{\mathrm{d}\xi} + \xi P_{2m}(\xi) = \frac{(-1)^m}{(2m-1)^{m-\frac{1}{2}}}\xi^{2m+1}. \tag{21}$$

Equation (21) is inconsistent, since the term of the highest order on the left-hand side is $\frac{2(-1)^m}{(2m+1)(2m)^{m-1/2}}\xi^{2m+1}$, which is clearly distinct from the term of the highest order on the right-hand side. □

Remark 3.7. Equations (12) and (16) can be easily generalized for the q-Fourier transform of q-Gaussians with nonzero means. Namely, let $\mu \neq 0$ be a real number, and

$$Y_{\mu,q}(\xi) = \int_{\mu-K}^{\mu+K} e_q^{-(x-\mu)^2+\mathrm{i}x\xi}\,\mathrm{d}x.$$

Then the associated functional differential equation for Y_{μ,q_n} with $q_n \in (0, 3)$ takes the form

$$2\sqrt{q_n}\frac{\mathrm{d}Y_{\mu,q_n}(\xi)}{\mathrm{d}\xi} + \xi Y_{\frac{\mu}{q_n},q_{n-2}}(\sqrt{q_n}\xi) - 2\mathrm{i}\mu\sqrt{q_n}\,Y_{\mu,q_n}(\xi) = I_{(0,1)}(q_n)r_{q_n}\xi^{\frac{1}{1-q_n}}. \tag{22}$$

3.2. Is the q-Fourier transform of a q-Gaussian a q′-Gaussian?

In this section we discuss a question important from applications point of view. Namely, we prove when the q-Fourier transform of a q-Gaussian is a q'-Gaussian with some index q' in $(-\infty, 3)$. With this aim we introduce the set of functions

$$\mathcal{G} = \bigcup_{q<3}\mathcal{G}_q, \qquad \text{where} \quad \mathcal{G}_q = \left\{f : f(x) = ae_q^{-\beta x^2}, a > 0, \beta > 0\right\}. \tag{23}$$

It follows from relationship (11) that if the q-Fourier transform $F_q[G_q(\beta, x)](\xi)$ of a q-Gaussian is a q'-Gaussian with some $q' \in (-\infty, 3)$, then $Y_q(\xi)$ must belong to $\mathcal{G}$. Therefore, we will study the existence of a solution of functional differential equations (12) and (16) in the set $\mathcal{G}$.

Theorem 3.8. *Let $1 \leqslant q < 3$ and $q_n, n \in \mathbb{N}_q$, be the sequence defined in (8). Then the functional differential equation*

$$2\sqrt{q_n}\frac{\mathrm{d}Y_{q_n}(\xi)}{\mathrm{d}\xi} + \xi Y_{q_{n-2}}(\sqrt{q_n}\xi) = 0, \qquad \xi \in \mathbb{R}, \tag{24}$$

has a unique solution $Y_{q_n}(\xi) \in \mathcal{G}$ satisfying the condition

$$Y_{q_n}(0) = C_{q_n}. \tag{25}$$

This solution is specifically

$$Y_{q_n}(\xi) = C_{q_n} e_{q_{n+1}}^{-\frac{3-q_n}{8}\xi^2}. \tag{26}$$

Proof. *Existence.* It follows immediately from representation (26) that $Y_{q_n}(0) = C_{q_n}$. Furthermore,

$$\frac{\mathrm{d}Y_{q_n}(\xi)}{\mathrm{d}\xi} = -\frac{1}{4}(3-q_n)C_{q_n}\xi\left(e_{q_{n+1}}^{-\frac{3-q_n}{8}\xi^2}\right)^{q_{n+1}}, \tag{27}$$

$$Y_{q_{n-2}}(\sqrt{q_n}\xi) = C_{q_{n-2}} e_{q_{n-1}}^{-q_n\frac{3-q_{n-2}}{8}\xi^2}. \tag{28}$$

Due to the equation $\left(e_q^y\right)^q = e_{2-1/q}^{qy}$ and part (i) of proposition 2.6, expression (27) can be rewritten as

$$\frac{\mathrm{d}Y_{q_n}(\xi)}{\mathrm{d}\xi} = -\frac{1}{4}(3-q_n)C_{q_n}\xi\, e_{q_{n-1}}^{-q_n\frac{3-q_{n-2}}{8}\xi^2}. \tag{29}$$

Substituting (28) and (29) into (24), we obtain

$$\left(-\sqrt{q_n}C_{q_n}\frac{3-q_n}{2} + C_{q_{n-2}}\right) e_{q_{n-1}}^{-\frac{q_n(3-q_n)}{8}\xi^2} = 0. \tag{30}$$

Now taking into account part (ii) of proposition 2.6 we conclude that $Y_{q_n}(\xi)$ in (26) satisfies (24).

Uniqueness. We note that $|\cos_q(x)| \leqslant 1$ for real x, if $q > 1$ (see [6]). This fact and corollary 2.3 imply the following estimate:

$$|Y_q(\xi)| = \left|\int_{-\infty}^{\infty} e_q^{-x^2+\mathrm{i}x\xi}\,\mathrm{d}x\right| \leqslant \int_{-\infty}^{\infty} e_q^{-x^2}\,\mathrm{d}x = C_q. \tag{31}$$

Assume that there are two solutions to problem (24)–(25), i.e. Y_{q_n} and $\tilde{Y}_{q_n}$. Then there difference $Z_{q_n}(\xi) = Y_{q_n}(\xi) - \tilde{Y}_{q_n}(\xi)$ also satisfies equation (24), and the condition $Z_{q_n}(0) = 0$. Now estimate (31) yields $Z_{q_n} \equiv 0$, which, in turn, implies $Y_{q_n} \equiv \tilde{Y}_{q_n}$. □

Corollary 3.9. *Let $q_n \geqslant 1$. Then*

$$F_{q_n}[G_{q_n}](\xi) = e_{q_{n+1}}^{-\frac{3-q_n}{8\beta^{2-q_n} C_{q_n}^{2(q_n-1)}}\xi^2}. \tag{32}$$

Remark 3.10. Representation (32) was obtained in [6] by the contour integration technique. The formula in (7) corresponds to the particular case $n = 0$ of (32).

Remark 3.11. If $q = 1$, then the Cauchy problem (24)–(25) reads

$$2\frac{\mathrm{d}Y_1(\xi)}{\mathrm{d}\xi} + \xi Y_1(\xi) = 0, \qquad Y_1(0) = \sqrt{\pi},$$

and its unique solution is $Y_1(\xi) = \sqrt{\pi}e^{-\xi^2/4}$. Besides corollary 3.9 we obtain

$$F\left[\frac{\sqrt{\beta}}{\sqrt{\pi}}e^{-\beta x^2}\right] = e^{-\frac{1}{4\beta}\xi^2}.$$

The density of the standard normal distribution corresponds to $\beta = 1/2$, giving the characteristic function of the classic Gaussian.

Theorem 3.12. *Let q_n, $n \in \mathbb{N}_q$, be a sequence defined in (8) with $q < 1$. Suppose that $q_n \neq m/(m+1)$, $m = 1, 2 \ldots$. Then the functional differential equation*

$$2\sqrt{q_n}\frac{\mathrm{d}Y_{q_n}(\xi)}{\mathrm{d}\xi} + \xi Y_{q_{n-2}}(\sqrt{q_n}\xi) = r_{q_n}\xi^{\frac{1}{1-q_n}}, \qquad \xi \in \mathbb{R}, \tag{33}$$

has no solution in $\mathcal{G}$.

Proof. We recall that if $q < 1$ and $n \in \mathbb{N}_q$, then $q_n < 1$. Further, we note that a function with compact support cannot solve equation (33). Since each function in $\mathcal{G}_q$ for $q < 1$ has a compact support, the solution of equation (33) cannot belong to $\mathcal{G}_q$, $q < 1$. Now assume that $Y_{q_n}(\xi) \in \mathcal{G}_{q'}$ and $Y_{q_{n-2}}(\xi) \in \mathcal{G}_{q''}$, with $q' > 1$ or $q'' > 1$ (the reader can easily verify that $q' \neq 1$ and $q'' \neq 1$). In accordance with definition (23),

$$Y_{q_n}(\xi) = ae_{q'}^{-b\xi^2} \qquad \text{and} \qquad Y_{q_{n-2}}(\xi) = Ae_{q''}^{-B\xi^2},$$

where a, b, A, B are some real positive numbers depending on q_n. Then, for equation (33) to be consistent,

$$\frac{2}{1-q'} - 1 = \frac{1}{1-q_n} \qquad \text{or} \qquad \frac{2}{1-q''} + 1 = \frac{1}{1-q_n}.$$

Solving these equations for q' and q'', one obtains $q' = \frac{q_n}{2-q_n}$ and $q'' = \frac{3q_n-2}{q_n}$. It follows $\max(q', q'') < 1$, since $q_n < 1$. This contradicts the assumption that $q' > 1$, or $q'' > 1$. □

Next, we consider the cases $q = \frac{1}{2}, \frac{2}{3}, \ldots, \frac{m}{m+1}, \ldots$, excluded from theorem 3.12. Direct computations show that in two specific cases, namely $q = 1/2$ and $q = 2/3$, $Y(q, \xi) \in \mathcal{G}_0$ considered on the positivity intervals. Indeed,

$$Y_{\frac{1}{2}}(\xi) = F_{\frac{1}{2}}\left[e_{\frac{1}{2}}^{-x^2}\right](\xi) = \frac{16\sqrt{2}}{15}\left(1 - \frac{5}{16}\xi^2\right), \tag{34}$$

which is nonnegative for $\xi \in [-4/\sqrt{5}, 4/\sqrt{5}]$. Therefore, on this interval we can associate it with an element of $\mathcal{G}_0$, writing $Y(1/2, \xi) = \frac{16\sqrt{2}}{15}e_0^{-(5/16)\xi^2} \in \mathcal{G}_0$. Similarly,

$$Y_{\frac{2}{3}}(\xi) = F_{\frac{2}{3}}\left[e_{\frac{2}{3}}^{-x^2}\right](\xi) = \frac{32\sqrt{3}}{35}\left(1 - \frac{7}{24}\xi^2\right) \in \mathcal{G}_0, \tag{35}$$

on the positivity interval $\left(-2\sqrt{\frac{6}{7}}, 2\sqrt{\frac{6}{7}}\right)$.

However, $Y(q, \xi)$ does not belong to $\mathcal{G}$ for any other value of $q = 3/4, 4/5, \ldots$. In order to show this first we derive an explicit form for $P_{m+1}(\xi) = Y_{m/(m+1)}(\xi)$. Recall that $P_{m+1}(\xi)$ is a polynomial of order $m + 1$ if $m + 1$ is even. Otherwise, it is a polynomial of order m.

Theorem 3.13. *Let* $q = m/(m+1), m = 1, 2, \ldots$ *. Then* $Y_q(\xi) = P_{m+1}(\xi)$ *has the representation*

$$P_{m+1}(\xi) = \sum_{k=0}^{[\frac{m+1}{2}]} (-1)^k \binom{m+1}{2k} (m+1)^{-k+\frac{1}{2}} B\left(k+\frac{1}{2}, m-2k+2\right) \xi^{2k}, \tag{36}$$

where $[x]$ *is the integer part of x, and B(a, b) is Euler's beta-function.*

Proof. Recall that if $q = \frac{m}{m+1}$, then $Y_q(\xi)$ has the form

$$Y_q(\xi) = P_{m+1}(\xi) = \int_{-\sqrt{m+1}}^{\sqrt{m+1}} \left(1 - \frac{1}{m+1}x^2 + \frac{1}{m+1}\mathrm{i}x\xi\right)^{m+1} \mathrm{d}x.$$

We have

$$P_{m+1}(\xi) = \sum_{k=0}^{m+1} \binom{m+1}{k} D_k(m) \frac{(\mathrm{i}\xi)^k}{(m+1)^k},$$

where

$$D_k(m) = \int_{-\sqrt{m+1}}^{\sqrt{m+1}} \left(1 - \frac{1}{m+1}x^2\right)^{m-k+1} x^k \,\mathrm{d}x.$$

It is not hard to verify that $D_k(m) = 0$ if k is odd and

$$D_{2k}(m) = (m+1)^{k+1/2} B(k+1/2, m-2k+2)$$

for $k = 0, \ldots, [\frac{m+1}{2}]$, which implies representation (36). □

Theorem 3.14. *Let* $q = m/(m+1), m = 3, 4, \ldots$ *. Then* $Y_q(\xi) \notin \mathcal{G}$.

Proof. It follows from representation (36) that the polynomial $Y_q(\xi) = P_{m+1}(\xi)$, with the first three (nonzero) terms indicated, reads

$$\begin{aligned} Y_q(\xi) &= D_0(m)\left[1 - (m+1)^2 \frac{B\left(\frac{3}{2}, m\right)}{B\left(\frac{1}{2}, m+2\right)}\xi^2 + \frac{m(m+1)^3}{2} \frac{B\left(\frac{5}{2}, m-2\right)}{B\left(\frac{1}{2}, m+2\right)}\xi^4 + \cdots\right] \\ &= D_0(m)\left[1 - \frac{2m+3}{8(m+1)}\xi^2 + \frac{(2m+3)(2m+1)}{8(m+1)^2}\xi^4 + \cdots\right], \end{aligned} \tag{37}$$

where

$$D_0(m) = C_{\frac{m}{m+1}} = \sqrt{m+1} B\left(\frac{1}{2}, m+2\right) = \frac{\sqrt{m+1}(m+1)!2^{m+2}}{(2m+3)!!}.$$

Now assume that $Y_q(\xi) \in \mathcal{G}_{q'}$ for some $q' < 3$. Then $1/(1-q') = (m+1)/2$, or $q' = (m-1)/(m+1)$. Therefore,

$$Y_q(\xi) = D_0(m)[1 - \beta(m)\xi^2]^{[\frac{m+1}{2}]},$$

where $\beta(m) > 0$ and $|\xi| \leqslant 1/\sqrt{\beta(m)}$. Applying the binomial formula and indicating the first three terms, one has

$$Y_q(\xi) = D_0(m)\left[1 - \frac{(m+1)\beta(m)}{2}\xi^2 + \frac{(m^2-1)[\beta(m)]^2}{8}\xi^4 + \cdots\right]. \tag{38}$$

Comparing the second and third terms in (37) and (38), one obtains contradictory relations

$$\beta(m) = \frac{2m+3}{4(m+1)^2}$$

and

$$[\beta(m)]^2 = \frac{(3m+3)(2m+1)}{(m-1)(m+1)^3} \neq \frac{(2m+3)^2}{16(m+1)^4} = [\beta(m)]^2, \qquad m = 3, 4, \ldots,$$

which completes the proof. □

Remark 3.15. Formula (36) for $q = 1/2$ and $q = 2/3$ gives

$$Y_{\frac{1}{2}}(\xi) = \frac{16\sqrt{2}}{15}\left(1 - \frac{5}{16}\xi^2\right) = \frac{16\sqrt{2}}{15} e_0^{-(5/16)\xi^2}, \qquad \xi \in \left[-\frac{4\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right],$$

and

$$Y_{\frac{2}{3}}(\xi) = \frac{32\sqrt{3}}{35}\left(1 - \frac{7}{24}\xi^2\right) = \frac{32\sqrt{3}}{35} e_0^{-\frac{7}{24}\xi^2}, \qquad \xi \in \left[-2\sqrt{\frac{7}{6}}, 2\sqrt{\frac{7}{6}}\right].$$

which coincide with (34) and (35), respectively. Both functions belong to $\mathcal{G}_0$.

Theorem 3.16. *Let $q < 0$. Then $Y_q(\xi) \notin \mathcal{G}$.*

Proof. Repeating calculations used in proofs of theorems 3.2 and 3.3, it is not hard to verify that the derivative of $Y_q(\xi)$ can be represented in the form

$$\frac{\mathrm{d}Y_q(\xi)}{\mathrm{d}\xi} = -\frac{\xi}{2}\int_{-K}^{K} \left(e_q^{-x^2+\mathrm{i}x\xi}\right)^q \mathrm{d}x + R_q \xi^{\frac{1}{1-q}}, \tag{39}$$

where

$$R_q = (1-q)^{\frac{1}{2(1-q)}} \sin\frac{\pi}{2(1-q)}. \tag{40}$$

We note that the condition $q < 0$ implies two statements important for further proof, namely $0 < \frac{1}{1-q} < 1$, and $R_q \neq 0$. Now assume that $Y_q \in \mathcal{G}_{q'}$ with some $q' \in (-\infty, 3)$. In other words, there are positive numbers a and b, such that $Y_q(\xi) = a e_{q'}^{-b\xi^2}$. Taking the first derivative of the latter, equating it to the right-hand side of (39), and dividing both sides by $\xi^{\frac{1}{1-q}}$ ($\xi \neq 0$), we obtain

$$\xi^{-\frac{q}{1-q}}\left[\frac{1}{2}\int_{-K}^{K} \left(e_q^{-x^2+\mathrm{i}x\xi}\right)^q \mathrm{d}x - 2ab\left(e_{q'}^{-b\xi^2}\right)^{q'}\right] = R_q, \tag{41}$$

which must be valid for all $\xi \in \mathbb{R}$. However, the left-hand side becomes arbitrarily small for small ξ, since $-\frac{q}{1-q} > 0$ and the expression in brackets has a finite limit at 0, while the right-hand side is nonzero constant. This contradiction completes the proof. □

4. Some applications to the porous medium equation

In this section we discuss some applications of the q-Fourier transform F_q to nonlinear models of partial differential equations. First we verify that the theorems proved in section 3 imply that F_q transfers a q-Gaussian into a q_1-Gaussian if $q \geqslant 1$, $q_1 = (1+q)/(3-q)$. Moreover, as shown in [11], the operator $F_q : G_q \to G_{q_1}$ for $q > 1$ is invertible. These two facts have been essentially used in [6, 10] for the proof of q-versions of the central limit theorem. Another application of F_q, as sketched hereunder, shows that it can be used for establishing a relation between the porous medium equation and a nonlinear ordinary differential equation (ODE) similar to the usual Fourier transform.

The classic Fourier transform reduces the Cauchy problem for linear partial differential equations of the form $u_t(t,x) = A(D_x)u(t,x) t > 0, x \in R^n, u(0,x) = \varphi(x), x \in R^n$,where

$D_x = (D_1, \ldots, D_n)$, $D_j = -\mathrm{i}\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, and $A(D_x)$ is an elliptic differential operator, to an associated linear ODE with the parameter $\xi \in R^n$. In the particular case of $n = 1$ and $A(D_x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ for the Fourier image $\hat{u}(t, \xi)$ of a solution $u(t, x)$, we have a dual differential equation

$$\hat{u}_t'(t, \xi) = -\xi^2 \hat{u}(t, x), \qquad \hat{u}(0, \xi) = \hat{\varphi}(\xi), \tag{42}$$

where $\xi \in R^1$ is a parameter. This instance corresponds to the Fokker–Planck equation without drift [15] for some classes of nonlinear differential equations.

We now demonstrate the similar role of F_q in the celebrated *porous medium equation* in the superdiffusion regime ubiquitously found in physical phenomena [17–21] (and references therein)[4]. Consider the following nonlinear diffusion equation with a singular diffusion coefficient:

$$\frac{\partial U}{\partial t} = (U^{1-q} U_x)_x, \qquad t > 0, \qquad x \in R^1, \qquad q > 1. \tag{43}$$

We look for a solution in the similarity set $G_q^* = \{U(t, x) : U(t, x) = t^a G_q(\beta; t^b x), a = a(q), b = b(q) \in R^1, \beta = \beta(q) > 0\}$, where a and β do not depend on t and x.

Proposition 4.1. *Suppose $U(t, x) \in G_q^*$ is a solution to equation (43). Then its q-Fourier transform $\hat{U}_q(t, \xi) = F_q[U(t, x)](\xi)$ satisfies the following nonlinear ordinary differential equation with the parameter ξ:*

$$(\hat{U}_q)_t' = -\frac{B(\beta, q)\xi^2}{t^{\frac{q-1}{3-q}}} (\hat{U}_q)^{q_1}, \qquad t > 0, \tag{44}$$

where $B(\beta, q) = \frac{2-q}{4\beta^{2-q} C_q^{q-1}}$ and $q_1 = \frac{1+q}{3-q}$.

Proof. Let $U \in G_q^*$ be a solution to (43), i.e. for some $a = a(q)$ and $\beta = \beta(q)$ it has the representation $U(t, x) = t^a G_q(\beta; t^a x)$. Then it follows from proposition 2.1 that

$$\begin{aligned}\hat{U}_q(t, \xi) &= F_q[U(t, x)](\xi) \\ &= F_q[G_q(\beta; x)]\left(\frac{\xi}{t^{a(2-q)}}\right) = \frac{1}{C_q} Y_q\left(\left(\frac{\sqrt{\beta}}{C_q}\right)^{q-1} \frac{\xi}{\sqrt{\beta} t^{a(2-q)}}\right),\end{aligned}$$

where $Y_q(\xi)$ is a solution to equation (24). Computing the derivative of $\hat{U}_q(t, x)$ in variable t, taking into account that $a = -1/(3-q)$ (see, e.g., [21]), and using equation (24), we obtain

$$(\hat{U}_q)_t = -\frac{2-q}{4\beta^{2-q} C_q^{2(q-1)}} \xi^2 (\hat{U}_q)^{q_1},$$

where $q_1 = (1+q)/(3-q)$. □

The inverse statement, given in the following formulation, is also true.

Proposition 4.2. *Suppose $V(t, \xi)$, $V(0, \xi) = 1$, is a solution to ODE with the parameter ξ*

$$V' = -\frac{B(\beta, q)\xi^2}{t^{\frac{q-1}{3-q}}} V^{q_1}, \qquad t > 0, \tag{45}$$

where $B(q, \beta)$ and q_1 are as in proposition 4.1. Then its inverse q-Fourier transform $U(t, x) = F_q^{-1}[V(t, \xi)](x)$ exists and satisfies equation (43).

[4] The monograph [21] contains different approaches to the solution of the porous medium equation.

Proof. By separation of variables of (45) one can verify that its solution

$$V(t,\xi) = e_{q_1}^{-\frac{3-q}{8\beta^{2-q}C_q^{q-1}}(\xi t^{\frac{2-q}{3-q}})}.$$

By theorem 0.6 of paper [11] the inverse q-Fourier transform for $V(t,\xi)$ exists, and by virtue of propositions 2.1 and 2.4 it has the representation

$$U(t,x) = \frac{1}{t^{\frac{1}{3-q}}} G_q\left(\beta(q); \frac{x}{t^{\frac{1}{3-q}}}\right), \qquad \text{where} \quad \beta(q) = \frac{1}{\left[2(3-q)C_q^{\frac{1}{q-1}}\right]^{\frac{2}{3-q}}}. \tag{46}$$

The latter is a solution to equation (43); see [21]. □

Note that if the initial condition is given in the form $U(0,x) = \delta(x)$ with the Dirac delta function, and $q = 1$, then we obtain (42) ($\hat{\varphi}(\xi) \equiv 1$), in which $\beta = 1/4$, $B(\beta, 1) = 4\beta = 1$.

In order to study price fluctuations in stock markets a stochastic process $X_t = \frac{\ln S(t+t_0)}{\ln S(t_0)}$ representing log-returns was introduced in [16]. Here $S(t)$ is the price at time t. X_t solves a stochastic differential equation $\mathrm{d}X_t = \tau\,\mathrm{d}t + \sigma\,\mathrm{d}\Omega_t$, where τ and σ are the drift and volatility coefficients respectively, and Ω_t is a solution to the Îto stochastic differential equation

$$\mathrm{d}\Omega_t = [P(\Omega_t)]^{\frac{1-q}{2}}\,\mathrm{d}B_t, \qquad t > t_0. \tag{47}$$

In this equation B_t is a Brownian motion, and P is a q-Gaussian distribution function. The corresponding Fokker–Planck-type equation in the case $\tau = 0, \sigma = 1$ reads

$$\frac{\partial V(x,t|x',t')}{\partial t} = ([V(x,t|x',t')]^{2-q})_{xx}, \tag{48}$$

which can easily be reduced to the form (43). From the financial applications point of view it is important to know the properties of the stochastic process X_t, since it can be considered as a q-alternative to the Brownian motion. One can effortlessly verify that if $U(t,x)$ is a solution to equation (43) for $t > 0$ with an initial condition $U(0,x) = f(x)$, then a solution $V(t,x), t > t'$, to the same equation (43) considered for $t > t'$ with an initial condition $V(t',x) = f(x)$ can be represented in the form $V(t,x) = U(t-t',x), t > t'$. It follows that X_t has stationary increments.

Concluding the discussion we note that solution (46) corresponds to the solution obtained from an ansatz [20] which is in accordance with the generalized central limit theorem presented in [6]. The method we have just presented for the model case can be implemented for other more general cases as well. For instance, the Fokker–Planck-type equation associated with a process X_t with constant drift $\tau = \mu \neq 0$, due to a term $-2\mathrm{i}\mu\sqrt{q_n}\,Y_{\mu,q_n}(\xi)$ in equation (22), has an additional drift term on the right-hand side of equation (48). We also note that with more routine calculations the method can be extended to the case of time-dependent drift and diffusion coefficients. We intend to present all the routine calculations in the general case of linear external forces in a separate paper.

5. Conclusion

Summarizing, we have the following general picture for the q-Fourier transform of q-Gaussians.

(1) The case $1 \leqslant q < 3$:

(1a) the q-Fourier transform acts as $F_q : \mathcal{G}_q \to \mathcal{G}_{q'}$;

(1b) the relation between q and q' is given by $q' = \frac{1+q}{3-q}$.

(2) The case $q = \frac{1}{2}$ or $q = \frac{2}{3}$: in this case the operator acts as $F_q : \mathcal{G}_q \to \mathcal{G}_0$. Relationship (1b) is failed.
(3) The case $q < 1$, but $q \neq \frac{1}{2}, \frac{2}{3}$: in this case (1a) is failed, in the sense that there is no q' such that the q-Fourier transform of a q-Gaussian would be a q'-Gaussian.

The lesson we have learnt from the above analysis is that the q-Fourier transform defined by formula (1) (or, the same, by formula (3)) is rich in content and applications if $q \in [1, 3)$. Its important applications in the case $q > 1$ are given in [6] for the proof of the q-central limit theorem, and in [9] for the classification of (q, α)-stable distributions. Another application of the operator F_q to the porous medium equation and related stochastic differential models is discussed in section 4 of the current paper. What concerns the case $q < 1$, the q-Fourier transform defined by formula (1) does not possess the properties valid in the case $1 \leqslant q < 3$. Therefore, the methods developed for $q \geqslant 1$ are not applicable in the case $q < 1$. The study of applications of the q-Fourier transform (or its alternatively defined version) to problems mentioned above, including the q-central limit theorem, remains a challenging open question in the case $q < 1$.

Acknowledgments

We acknowledge C Tsallis for several comments on the subjects mentioned in this paper. We also acknowledge thoughtful remarks by anonymous referees which significantly improved the text.

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